4825. *Proposed by Ovidiu Furdui and Alina Sîntămărian.*

Let $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$, $n \geq 1$. Calculate

$$\sum_{n=1}^{\infty} \frac{O_n}{n(n+1)}.$$

Solution 3, by Devis Alvarado, G.C. Greubel and the Missouri State University Problem Solving Group, done independently.

Since

$$O_n = \int_0^1 \frac{1 - x^{2n}}{1 - x^2}\, dx,$$

we get

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{O_n}{n(n+1)} &= \int_0^1 \left[\frac{1}{1-x^2} \sum_{n=1}^{\infty} \frac{1 - x^{2n}}{n(n+1)} \right] dx \\
&= \int_0^1 \left[\frac{1}{1-x^2} \left(1 - \sum_{n=1}^{\infty} \left(\frac{x^{2n}}{n} - \frac{x^{2n}}{n+1} \right) \right) \right] dx \\
&= \int_0^1 \left[\frac{1}{1-x^2} \left(1 - \left(-\log(1-x^2) + \frac{1}{x^2} \log(1-x^2) + 1 \right) \right) \right] dx \\
&= -\int_0^1 \frac{\log(1-x^2)}{x^2}\, dx \\
&= \lim_{a \uparrow 1} \int_0^a \frac{-\log(1-x^2)}{x^2}\, dx.
\end{aligned}$$

Integrating by parts (with $u = -\log(1-x^2), dv = x^{-2}dx$), we find that

$$\begin{aligned}
\int_0^a \frac{-\log(1-x^2)}{x^2}\, dx &= \left[\frac{\log(1-x^2)}{x} \right]_0^a + \int_0^a \frac{2}{1-x^2}\, dx \\
&= \frac{\log(1-a^2)}{a} + \int_0^a \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx \\
&= \left(\frac{\log(1-a) + \log(1+a)}{a} \right) + \log(1+a) - \log(1-a) \\
&= \frac{(1-a)\log(1-a)}{a} + \frac{(1+a)\log(1+a)}{a}.
\end{aligned}$$

Letting a increase to 1 reveals that

$$-\int_0^1 \frac{\log(1-x^2)}{x^2}\, dx = 2\log 2.$$